On the possibility to view the fluctuations of velocity and temperature during a laminar-turbulent transition as fractal time series

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Abstract. The present paper discusses a novel technique – dynamic analysis of multi-fractal time series – that may become a useful tool for improving the understanding of transition from laminar to turbulent flow regimes. It is proposed to view records of velocity and temperature fluctuations as fractal time series, whose measures, such as fractal dimensions, the Jeffreys divergence measure, and the Hurst exponent, themselves vary in time. The validation tests show that the variations of the fractal measures may be used for quantifying the transition between the flow regimes.

1 Introduction

In the course of more than a century, the problem of transition to turbulence causes great interest of researchers. This interest is due to, firstly, the necessity of solving practical problems, related,
for instance, with the task to control the boundary layer, in order to decrease the drag of aircrafts and vessels. Secondly, the study of transition to turbulence is an integral part of a more fundamental problem, namely: the description of the very phenomenon of turbulence.

In the end of the nineteenth century, Reynolds [1] and Rayleigh [2] hypothesized that the reason of transition from laminar flows to turbulent ones is the instability of the former. In other words, an increase of the intensity of waves within an instable boundary layer leads to the destruction of laminar flows. In 1924, Heisenberg [3] laid the foundation for developing a linear theory of hydrodynamic instability. First computations of the boundary layer stability were performed much later – between the end of the 20-th and the beginning of the 30-th – by Tollmien [4] and Schlichting [5-7]. Later on, Taylor [8] proposed another hypothesis, according to which transition to turbulence is caused by fluctuations within the outer flow that lead to the local separations of the boundary layer. Until the 40-th, doubts about the validity of the stability theory prevailed, for the experimental evidence of that period was in favor of the Taylor hypothesis. In particular, the experiments by Dryden [9-10], who applied a novel for that time technique: measurements by the use of thermo-anemometer, were supportive of the Taylor hypothesis.

In 1948, for the first time, Schubauer and Skramstad [11] found out how fluctuations within the boundary layer play the key role in the destruction of the laminar flow regime.

If one speaks about a rigorous theory of the laminar-turbulent transition, then such an adequate formal theory, in the first place, has to allow the description of the evolution of instability: its appearance, development, the observable destruction of the laminar flow, and the final transition to a fully developed turbulent flow. Solving such a problem involves great mathematical difficulties. In fact, in spite of all efforts, until today there exist no fundamental theory of transition, but there are experiments and correlations which try to predict the final onset of fully turbulent flow.

At present, there are three principal concepts used for transition modeling. The first concept is the application of the so-called low Reynolds number turbulence models. These models, however, are not suitable to predict transition adequately. This is because the calibration of the damping functions, involved in the mentioned models, is based on reproducing the viscous sub-layer behavior, but not on predicting the laminar-turbulent transition.

The second concept is the use of experimental correlations. The correlations usually relate the free-stream turbulence intensity and the local pressure gradient to the Reynolds number based on the transition momentum thickness. A typical example is the Abu-Ghannam and Shaw correlation [12], which is based on a large number of experimental observations. While this method proves sufficiently accurate, it poses numerical and programming challenges in the codes for the Navier-Stokes equations.

For classical correlation-based transition models, it is necessary to compare the actual momentum-thickness Reynolds numbers with the transition value from the correlation. This is not an easy task in a numerical simulation. The difficulties, associated with non-local formulations, arise from the use of modern numerical methods that are based on unstructured grids and massive parallel execution. Unstructured grids do not easily provide the infrastructure needed to integrate global boundary layer parameters, because the grid-lines normal to the surface cannot be easily identified. In the case of a general parallelized code, the boundary layer can be split between different CPUs making the integration tedious to perform in parallel. The use of correlation-based transition criteria is therefore incompatible with modern numerical codes.

Recently a number of attempts at developing a compatible model appeared [13]. The predictive Menter-Langtry $\gamma-\theta$ laminar-turbulent transition model $TM$ has been extended to aeronautical flows [14, 15]. The model is based on two transport equations that use local information only and is there-
fore fully compatible with modern numerical codes. The proposed transport equations do not attempt to model the physics of the transition process (unlike, for instance, turbulence models), but form a framework for the implementation of correlation-based models into general-purpose numerical methods. In order to distinguish this concept from the physics-based transport models, it is named LCTM – Local Correlation-Based Transition Modeling.

The third concept is based on the boundary layer stability theory [16-18]. In investigating the stability of laminar flows, the motion is decomposed into the basic flow, whose stability is to be examined, and a perturbation motion. Only those perturbations which are consistent with the hydrodynamic equations are allowed and the progression in time of the disturbance is traced. This is the method of small disturbances.

Stability analysis gives the position of the instability point. However, it takes some time until perturbations would be amplified enough to produce turbulence. Therefore, the observed position of the laminar-turbulent transition is always further downstream than the theoretically computed limit of stability.

The so-called $e^N$ method is based on a local linear stability theory and the parallel flow assumption, in order to calculate the growth of the disturbance amplitude from the boundary layer neutral point to the transition location [19]. Once the disturbance amplitude ratio $e^N$ exceeds the limiting factor, the transition is assumed to start. The limiting factor is not known in advance and must be determined by calibration in wind tunnels or by flight tests; hence, the $e^N$ approach is considered as a semi-empirical method.

For isolated airfoils, the $e^N$ method has been shown to produce very good transition predictions compared to wind tunnel measurements [20]. However, there remain some formidable barriers towards using the $e^N$ method in general applications.

The first is that since the $e^N$ method is based on a linear stability theory, it cannot predict transition due to non-linear effects such as high free-stream turbulence or surface roughness. In addition, typical industrial solutions to the Navier-Stokes equations are not accurate enough to evaluate the stability equations. As a result, the Navier-Stokes solution must be coupled to an accurate boundary layer code [20]. Finally, the need to track the growth of the disturbance amplitude ratio along the streamline results in a significant issue for three-dimensional flows, where the streamline direction is not aligned with the grid.

Because of this limitation, one is not aware of any instance in the open literature, where the $e^N$ method has been successfully applied to predict transition on a fully 3-dimensional flow. However, the recent development of high-order methods could lead to the possibility of modifying the $e^N$ method in such a way that it will become compatible with the general-purpose numerical methods and could be applied to arbitrary geometries.

From the above discussion, it becomes clear that it is worth exploring the possibility of developing alternative easy-to-use models and methods that allow one to predict and quantify the process of transition.

The present paper is on the possibility to view records of velocity (and possibly temperature) fluctuations as multi-fractal time series with the purpose to predict and quantify laminar-turbulent transitions. Application of ideas from fractal geometry to describe turbulent behavior is by no means new [21, 22]. In most of the works that explore this idea, however, spatial “fractalness” of turbulence is emphasized. This paper, in contrast, discusses the method that allows one to study how fractal measures (e. g., fractal dimensions of the velocity fluctuations) vary in time. If successful, such a transient method should be able to forecast and quantify the onset of the turbulent flow regime.
2 Mathematical Analysis

Velocity variations in time are usually recorded as fluctuations of electric potential by a probe inserted into the stream (e. g., hot-wire anemometer). Hence, such records can be viewed as temporal sequences (time series). Signals of such a kind can be generated by different means and, for this reason, be of different nature: random noise, chaotic process, or bios [23].

One common practice to distinguish among possible classes of time series is to determine their so-called correlation dimension. It has been shown that the correlation dimension of a velocity variation assumes different values depending on the flow regime (e. g., laminar, turbulent, or intermittent) [22]. The correlation dimension, however, belongs to an infinite family of fractal dimensions [24]. Hence, there is a hope that the use of the whole family of fractal dimensions may prove to be advantageous in comparison with using only some of these dimensions.

2.1 Fractal dimensions

The concept of generalized entropy of a probability distribution was introduced by Alfred Rényi [25]. Based on the moments of order \( q \) of the probability \( p_i \), Rényi obtained the following expression for entropy

\[
S_q = \frac{1}{1-q} \log \sum_{i=1}^{N} p_i^q
\]

where \( q \) is not necessarily an integer and \( \log \) denotes \( \log_2 \). Note that for \( q \to 1 \), Eq. (2.1) yields the well-known entropy of a discrete probability distribution [26]

\[
S_1 = - \sum_{i=1}^{N} p_i \log p_i
\]

The probability distribution of a given time series can be recovered by the following procedure. The total range of the signal is divided into \( N \) bins such that

\[
N = \frac{V_{\text{max}} - V_{\text{min}}}{\delta V}
\]

where \( V_{\text{max}} \) and \( V_{\text{min}} \) are the maximum and the minimum values of the signal achieved in the course of measurements, respectively; \( \delta V \) represents the sensitivity (uncertainty) of the measuring device. The probability that the signal falls into the \( i \)--th bin of size \( \delta V \) is computed as

\[
p_i = \lim_{N \to \infty} \frac{N_i}{N}
\]

where \( N_i \) equals the number of times the signal falls into the \( i \)--th bin. On the other hand, in the case of a time series, the same probability can be found from the ergodic theorem, that is

\[
p_i = \lim_{T \to \infty} \frac{t_i}{T}
\]

where \( t_i \) is the time spent by the signal in the \( i \)--th bin during the total time span of measurements \( T \).

Further, the generalized fractal dimensions of a given time series with the known probability distribution are defined as
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\[ D_q = \lim_{\delta V \to 0} \frac{1}{q-1} \log \sum_{i=1}^{N} p_i^q \] (2.6)

where the parameter \( q \) ranges from \(-\infty\) to \(+\infty\). Note that for a self-similar fractal time series with equal probabilities \( p_i = 1/N \), the definition Eq. (2.6) yields \( D_q = D_0 \) for all values of \( q \) [27]. Note also that for a constant signal, all probabilities except one become equal to zero, whereas the remaining probability value equals unity. As a result, for a constant signal, \( D_q = D_0 = 0 \), which geometrically is equivalent to a single point on the timeline. The fractal dimension

\[ D_0 = -\frac{\log N}{\log \delta V} \] (2.7)

is nothing else than the Hausdorff-Besicovitch dimension [28].

The correlation dimension, mentioned previously, is the fractal dimension with \( q = 2 \).

As \( q \to 1 \), Eq. (2.6) yields the so-called information dimension

\[ D_1 = \lim_{\delta V \to 0} \frac{-\sum_{i=1}^{N} p_i \log p_i}{\log(1/\delta V)} \] (2.8)

where the numerator is Shannon’s entropy given by Eq. (2.2).

Note also that

\[ D_\infty = \lim_{\delta V \to 0} \frac{\log p_{\max}}{\log \delta V} \] (2.9)

and

\[ D_{-\infty} = \lim_{\delta V \to 0} \frac{\log p_{\min}}{\log \delta V} \] (2.10)

such that \( D_{-\infty} \geq D_\infty \). In general, if \( a < b \), \( D_a \geq D_b \), such that \( D_q \) is a monotone non-increasing function of \( q \) [27]. For a given time series (“signal”), the function \( D_q \), corresponding to the probability distribution of this signal, is called the fractal spectrum. Such a name is well-justified, because the fractal spectrum provides information about both frequencies and amplitudes of the signal. Indeed, for two probability distributions, a larger value of a fractal dimension of a given order corresponds to the presence of more pronounced spikes (sharper spikes, less expected values of the signal) than in the signal for which the value of the fractal dimension of the same order is less. Furthermore, signals with a wider range of fractal dimensions, \( D_{-\infty} - D_\infty \), can be termed more fractal than signals whose range of fractal dimensions is narrower, so that signals with the zero range are self-similar fractals. In other words, the range of a fractal spectrum is a value associated with the range of frequencies in the signal.

Now, if the unexpectedness of an event is defined as the inverse of the probability of this event, then steeper spectra correspond to the signals in which unexpected values are more dominant, whereas flatter spectra represent those signals in which less unexpectedness occurs.

2.2 Jeffreys divergence measure

The Jeffreys divergence measure is defined as

\[ J(u, v) = \sum_i u_i \log \left( \frac{u_i}{v_i} \right) + \sum_i v_i \log \left( \frac{v_i}{u_i} \right) \] (2.11)
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and gives the distance between two distributions \( u \) and \( v \). This measure is non-negative, and it can be considered as a measure of the dissimilarity between two probability distributions. In this work, it is used to determine the distance between two parts of a single record of velocity or temperature variations.

It is worth noting here that the Jeffreys divergence measure is proportional to the entropy growth between the distributions (compare (2.11) and (2.2)).

2.3 Hurst exponent

In general, the mean square value of a random variable does not increase linearly in time. Such a random-walk process is called anomalous diffusion, in order to distinguish it from the classical Brownian motion (normal diffusion).

The second moment of the random walk variable is

\[
\langle X^2(t) \rangle \propto t^{2H}
\]

where \( H \) is called the Hurst exponent [29].

When \( H \neq 1/2 \), equation (2.12) depicts anomalous diffusion. When \( 1/2 < H \leq 1 \) the random walk is persistent, meaning that a walker prefers to continue in the direction of motion rather than changing direction, and therefore the diffusion is faster than normal (super-diffusion, turbulent processes). When \( 0 \leq H < 1/2 \) the random walk is anti-persistent, meaning that a walker prefers to change direction with each step rather than to continue in the same direction, and therefore the diffusion is slower than normal. Finally, normal diffusion occurs when \( H = 1/2 \) and is usually referred to as Brownian motion.

3 Dynamic Analysis of Fractal Time Series

This section is devoted to dynamic analysis of fractal time series that is performed by the use of newly developed software Fractal Time Series Analyzer (FRATSAN).

For a data sample, whose total duration is \( N \), a sliding window of size \( n < N \) is chosen. The size of the sliding window must be large enough, to be statistically meaningful, on the one hand; and, on the other hand, the values of fractal measures (e.g., fractal dimensions) computed for the sliding window should not depend on the window size. In order to satisfy both conditions, a preliminary test is conducted before processing any given set of data. In practice, for most of long data samples, the size of the sliding window is between 4500 and 5000 data points.

As the size of the sliding window is chosen, the set of values of fractal dimensions and the Jeffreys divergence measure are computed for the sliding window positioned at the first data element of the total sample of data. Then the sliding window is positioned at the second element of the data sample and computations of all the fractal measures are repeated. By moving the sliding window through all the elements of the data set, the temporal (transient/dynamic) behavior of the fractal measures is recovered.

The Hurst exponent is computed as a function of time, \( H(t) \), for the entire set of data, past with respect to any given moment \( t \). The reason for not using the sliding window in case of computing the Hurst exponent can be understood from the physical interpretation of this parameter. Indeed, it can be shown that the process of anomalous diffusion can be modeled by
where $t$ denotes time and $\mathcal{J}$ is an operator on a Banach space [30]. The fractional derivative in (3.1) is understood in the Riemann-Liouville sense, that is,

$$\frac{\partial^\alpha f}{\partial t^\alpha} = \frac{1}{\Gamma(-\alpha)} \int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{\alpha+1}}, \quad \text{Re}(\alpha) < 0$$

where $\Gamma(\alpha)$ is the Gamma function.

Hence, the Hurst exponent classifies and quantifies the influence of the past history. Small values of $H$ correspond to a strong influence of the past history (laminar processes), while its large values represent processes with short or no memory.

Therefore, if the Hurst exponent were computed for a sliding window only, the past history available for analysis would not be used fully and, consequently, one might end up with a wrong value of the Hurst exponent. In order to avoid this, for any given $t$, the Hurst exponent was computed from (2.12), using $\langle X^2(t) \rangle$ of the entire data set preceding the given moment $t$.

4 Validation Tests

In order to validate the method, two sets of data with known fractal characteristics were generated. Figure 1 shows the first set of data that mimics the intermittent increase of the signal amplitude. The set consists of five packets (subsets) of equal size of 10,000 data points. While the first packet contains 90 per cent of low-amplitude signal and 10 per cent of high-amplitude signal, the share of the high-amplitude signal is increased to 30 per cent within the second subset, 50 per cent within the third subset, 70 per cent in the fourth one, and 90 per cent in the last packet, respectively. Note that the signal shown in Fig. 1 is much simpler than signals recorded in real processes of transition. Yet, it is hardly possible to describe even this “simple” signal by means of analytic functions.
The corresponding time evolution of the three fractal dimensions of the given set, namely $D_{-\infty}$, $D_0$ and $D_{+\infty}$ is depicted in Fig. 2. The fact that all the three values differ proves that the signal depicted in Fig. 1 is a multi-fractal time series.

![Fig. 2 Time evolution of the three fractal dimensions for the first test data set.](image)

As can be seen from the figure, the fractal dimensions exhibit much smoother behavior than the signal itself and hence, it is much easier and faster to approximate them by analytic functions, e.g. Fourier series. Moreover, changes of the fractal dimensions’ values clearly follow the changes within the signal itself.

Figure 3 shows the time evolution of the corresponding Jeffreys divergence measure, while the time evolution of the average value of the divergence measure is shown in Figure 4.

![Fig. 3 Time evolution of the Jeffreys divergence measure for the first test data set.](image)

Like the changes of the fractal dimensions’ values, the behavior of the Jeffreys divergence measure allows one to clearly distinguish among different subsets. Yet, in addition, one can see that the probability distribution of each consequent packet is “further” away from the previous one and, hence,
each subsequent packet “moves” away from the initial subset. At the same time, it shows that each packet has some memory about the entire past history, although this memory decreases as time goes on. This issue will be further discussed when the value of the Hurst exponent is analyzed.

![Figure 4](image-url)  
**Fig. 4** Time evolution of the averaged Jeffrey divergence measure for the first test data set.

The behavior of the averaged value of the Jeffrey divergence measure is even more telling. It can be clearly seen that the distance between the probability distributions characterizing the subsets increases almost exponentially with time. The function depicted in Fig. 4 is very smooth and, therefore, allows one to easily quantify how the process in question evolves.

Finally, Figure 5 shows the time evolution of the corresponding Hurst exponent. As expected for a well-ordered data set, the value of the Hurst exponents is not large and is well below 1/2. It further proves that the behavior of the values within the set is well-predictable (this should be the case, because a very clear and simple algorithm has been used to generate the set), the signal possesses a strong memory about its past history and is anti-persistent. Indeed, the maximum deviation practically does not vary from packet to packet.

![Figure 5](image-url)  
**Fig. 5** Time evolution of the Hurst exponent for the first test data set.
However, the value of the Hurst exponent is by no means constant, it slowly but steadily increases with time. This is another argument in favor of the signal’s slowly losing its memory about the past history. (Recall that signals with $H = 1/2$ have no memory at all.)

Figure 6 shows the second set of data that mimics the laminar-to-turbulent transition process. In the same way as the first data set, the second set consists of five packets (subsets) of equal size of 10,000 data points. While the first packet contains 90 per cent of low-amplitude signal and 10 per cent of high-amplitude signal, the share of the high-amplitude signal is increased to 30 per cent within the second subset, 50 per cent within the third subset, 70 per cent in the fourth one, and 90 per cent in the last packet, respectively. Unlike in the first data set, not only is the share of the high-amplitude signal increased in each subset, but also the value of the amplitude increases twice for each consecutive packet, while the value of amplitude within the low-amplitude part remains the same for all the packets.

The corresponding time evolution of the three fractal dimensions of the given set, namely $D_{-\infty}$, $D_0$ and $D_{+\infty}$ is depicted in Fig. 7. The fact that all the three values differ proves that the signal depicted in Fig. 6 is a multi-fractal time series.

As can be seen from the figure, the fractal dimensions exhibit much smoother behavior than the signal itself and hence, it is much easier and faster to approximate them by analytic functions, e.g. Fourier series. Moreover, changes of the fractal dimensions’ values clearly follow the changes within the signal itself. Note, however, that in comparison with Fig. 2, where the differences between the values of the fractal dimensions remain nearly constant, in Fig. 7, on can clearly see that the values of $D_0$ and $D_{+\infty}$ approach the value of $D_{-\infty}$. This is very telling, because, by definition, the values of $D_0$ and $D_{+\infty}$ can only be smaller or equal to $D_{-\infty}$.

Figure 8 shows the time evolution of the corresponding Jeffreys divergence measure, while the time evolution of the average value of the divergence measure is shown in Figure 9.

Like the changes of the fractal dimensions’ values, the behavior of the Jeffreys divergence measure allows one to clearly distinguish among different subsets. Yet, in addition, one can see that the probability distribution of each consequent packet is “further” away from the previous one and, hence, each subsequent packet “moves” away from the initial subset. At the same time, it shows that each packet has some memory about the entire past history, although this memory decreases as time goes
on. It is worth noting here that the values of the Jeffreys divergence measure are about twice larger than those shown in Fig. 3. Hence, they well reflect the increase in the high-amplitude part within each consecutive packet.

It must be stressed here that the functions shown in Fig. 3 and Fig. 8 are known as the “Devil’s staircase”. In contrast to the self-similar measures that characterize simple fractals, the “Devil’s staircase” – known as an intermittent function – occurs in the case of multi-fractals only [22, 27]. This is another evidence of the fact that both data sets, used in this study, are intermittent random processes and can be viewed as multi-fractal time series.

As in the first data set, the behavior of the averaged value of the Jeffreys divergence measure is even more telling. It can be clearly seen that the distance between the probability distributions characterizing the subsets increases almost exponentially with time. The function depicted in Fig. 9 is again very smooth and, therefore, allows one to easily quantify how the process in question evolves. As could be already anticipated from Fig. 8, the averaged values of the Jeffreys divergence measure are twice larger than those depicted in Fig. 4.
Finally, Figure 10 shows the time evolution of the corresponding Hurst exponent. In comparison with the first data set, where the value of the Hurst exponent increases very slowly and barely reaches 0.1 over the entire history, the value shown in Fig. 10 increases much faster and, although it never exceeds 0.5, it almost reaches 0.4 over the entire record. This is evidence that the second set loses its predictability much faster than the first set of data, making the record within the last packet almost deprived of predictability.

5 Conclusions

It has been demonstrated that the dynamic analysis of random processes, seen as fractal time series may become a useful tool for improving the understanding of such random processes, in which a transition between different modes of random walks takes place. Among physical processes of such a
kind, the least understood and, hence, the most interesting is the process of transition from the laminar flow to the turbulent one. In such a process, the dependence of the past history varies with time, as do probability distributions of the fluctuating values of velocities and temperature.

There is a hope that the way of analysis, introduced in this work, may become useful for quantifying the process in question and, by doing so, developing better mathematical and physical models of the process of transition.

6 Acknowledgements

The work presented in this paper has been supported by the Czech Science Foundation project No. 103/07/0136 and by the Ministry of Defense project No. FVT 0000404.

The authors are grateful to the School of Mechanical and Aerospace Engineering at Nanyang Technological University (Singapore) for their support of faculty exchange in the course of writing this paper.

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